

FANO 5-FOLDS WITH NEF TANGENT BUNDLES AND PICARD NUMBERS GREATER THAN ONE

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ABSTRACT. We prove that smooth Fano 5-folds with nef tangent bundles and Picard numbers greater than one are rational homogeneous manifolds.

1. INTRODUCTION

Characterization problems of special projective manifolds in terms of positivity properties of the tangent bundle have been considered by several authors. One of the most important results is S. Mori's solution of the Hartshorne-Frankel conjecture [14]: a projective manifold with ample tangent bundle is a projective space.

As a generalization of Mori's theorem, F. Campana and T. Peternell [4] proposed to study complex projective manifolds with nef tangent bundles and gave the classification in case of dimension 3. After that, a structure theorem of such manifolds in arbitrary dimension was provided by J. P. Demailly, T. Peternell and M. Schneider [7]: a projective (or more generally, compact Kähler) manifold X with nef tangent bundle admits a finite étale cover $\tilde{X} \rightarrow X$ such that the Albanese map $\tilde{X} \rightarrow \text{Alb}(\tilde{X})$ is a smooth morphism whose fibers are Fano manifolds with nef tangent bundles.

Hence, we obtain the complete picture of projective manifolds with nef tangent bundles if the following conjecture due to Campana and Peternell is solved:

Conjecture 1.1 ([4]). *A Fano manifold X with nef tangent bundle is rational homogeneous.*

By the classification theory of Fano manifolds, one can check that this conjecture holds when $\dim X \leq 3$. Furthermore, Campana and Peternell [5] gave an affirmative answer when $\dim X = 4$ and the Picard number $\rho_X > 1$. After that, via the works of [6], [12] and [13], the case when $\dim X = 4$ was finally completed by J. M. Hwang [9]. However this conjecture remains open in $\dim X \geq 5$. Our main purpose of this article is to treat the case when $\dim X = 5$ and $\rho_X > 1$.

Theorem 1.2 (=Theorem 4.1). *Let X be a complex Fano manifold of dimension 5 with nef tangent bundle and Picard number $\rho_X > 1$. Then X is a rational homogeneous manifold.*

The proof proceeds as follows. Let X be a Fano 5-fold with nef tangent bundle of $\rho_X > 1$. For any contraction $f : X \rightarrow Y$ of an extremal ray, f is smooth, and Y and the fibers X_y are Fano manifolds with nef tangent bundles (Theorem 3.4). Furthermore, we see that $\rho_{X_y} = 1$. Since Conjecture 1.1 holds for Fano manifolds of dimension ≤ 4 , it is easy to see that X is a holomorphic fiber bundle over a

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rational homogeneous manifold Y whose fibers are projective spaces or quadrics (Lemma 4.2). Since $\rho_X > 1$, X admits at least two different fiber bundle structures. Studying these bundle structures, we get the complete classification.

This paper is organized as follows: In Section 2, we recall some known results on Fano manifolds. Section 3 is dedicated to study properties of Fano manifolds with nef tangent bundles. Furthermore, we shall determine if some concrete examples of Fano manifolds with projective bundle structures have nef tangent bundles. In Section 4, we prove our main result Theorem 1.2. In the final section, we deal with Fano 5-folds with nef tangent bundles of $\rho = 1$.

In this paper, we use notation as in [8] and every point on a variety we deal with is a closed point. Denote the m times product of \mathbb{P}^n by $(\mathbb{P}^n)^m$. We work over the field of complex numbers.

2. KNOWN RESULTS ON FANO MANIFOLDS

A *Fano manifold* means a projective manifold X with ample anticanonical divisor $-K_X$. For a Fano manifold X , the *pseudoindex* is defined as the minimum i_X of the anticanonical degrees of rational curves on X .

Given a projective manifold X , we denote by $N_1(X)$ the space of 1-cycles with real coefficients modulo numerical equivalence. The dimension of $N_1(X)$ is the Picard number ρ_X of X . The convex cone of effective 1-cycles in $N_1(X)$ is denoted by $NE(X)$. By the Contraction Theorem, given a K_X -negative extremal ray R of the Kleiman-Mori cone $\overline{NE}(X)$, we obtain the contraction of the extremal ray $\varphi_R : X \rightarrow Y$. We say that φ_R is of *fiber type* if $\dim X > \dim Y$, otherwise it is of *birational type*.

Proposition 2.1 ([3, Lemma 3.3, Remark 3.7]). *Let X be a Fano manifold, $f : X \rightarrow Y$ a contraction of an extremal ray of fiber type, and X_y a fiber of f . Suppose that f is smooth. Then X_y is a Fano manifold of $\rho_{X_y} = 1$.*

Proof. We show that $\rho_{X_y} = 1$. Let $N_1(X/Y)$ be the vector space generated by irreducible curves which are contracted by f , modulo numerical equivalence. Let $NE(X/Y) \subset N_1(X/Y)$ be the cone generated by the class of effective curves contracted by f . From the (relative) Cone Theorem, it turns out that

$$NE(X/Y) = \sum_{i: \text{finite}} \mathbb{R}_{\geq 0}[C_i].$$

For any fiber X_y and the natural inclusion $i : X_y \subset X$, consider the linear map

$$i_* : N_1(X_y) \rightarrow N_1(X).$$

We see that $i_*(NE(X_y)) \subset NE(X/Y)$. Then [24, Proposition 1.3] tells us that $\text{Locus}(\mathbb{R}_{\geq 0}[C_i])$ dominates Y via f , where $\text{Locus}(\mathbb{R}_{\geq 0}[C_i])$ is the locus of curves whose classes are in $\mathbb{R}_{\geq 0}[C_i]$. This implies that $\mathbb{R}_{\geq 0}[C_i] \subset i_*(NE(X_y))$ for any $y \in Y$. Thus, we have

$$i_*(NE(X_y)) = NE(X/Y)$$

for any $y \in Y$. So we get

$$i_*(N_1(X_y)) = N_1(X/Y).$$

In particular, it turns out that

$$\dim i_*(N_1(X_y)) = \rho_X - \rho_Y.$$

On the other hand, we have $H^2(X, \mathbb{Q}) \cong \text{Pic}(X) \otimes \mathbb{Q}$ and $H^2(X_y, \mathbb{Q}) \cong \text{Pic}(X_y) \otimes \mathbb{Q}$, since X and X_y are Fano manifolds. Remark that $\dim i_*(N_1(X_y))$ is equal to the dimension of the image of

$$\text{Pic}(X) \otimes \mathbb{Q} \cong H^2(X, \mathbb{Q}) \rightarrow H^2(X_y, \mathbb{Q}) \cong \text{Pic}(X_y) \otimes \mathbb{Q}.$$

Furthermore, it is equal to the dimension of the image of the linear subspace of $H^2(X_y, \mathbb{Q})$ which is invariant for the monodromy action of $\pi_1(Y, y)$ (see for example [23, Theorem 4.18]). Remark that an image of a Fano manifold by a smooth morphism is again Fano (see [11, Corollary 2.9]). So Y is simply-connected. Hence

$$\dim i_*(N_1(X_y)) = \dim H^2(X_y, \mathbb{Q}) = \rho_{X_y}.$$

As a consequence, we have $\rho_{X_y} = \rho_X - \rho_Y = 1$. □

Proposition 2.2 ([15, Lemma 2.13]). *Let $f : X \rightarrow Y$ be a smooth morphism from a Fano manifold X . If every fiber of f is \mathbb{P}^d , then there exists a rank $(d+1)$ vector bundle \mathcal{E} on Y such that $X = \mathbb{P}_Y(\mathcal{E})$.*

Proof. The same argument as in Proposition 2.1 implies that Y is simply-connected. Moreover $H^*(\mathbb{P}^d, \mathbb{Z}) \cong \mathbb{Z}[x]/(x^{d+1})$, where x is the first Chern class of the hyperplane bundle. Let X_y be a fiber of f . By applying Lemma 2.3 below, the restriction map $H^2(X, \mathbb{Z}) \rightarrow H^2(X_y, \mathbb{Z})$ is surjective. Furthermore, since X is a Fano manifold, we see that $\text{Pic}(X) \cong H^2(X, \mathbb{Z})$. Hence there exists a line bundle $L \in \text{Pic}(X)$ such that $\mathcal{O}_X(L)|_{X_y} = \mathcal{O}_{\mathbb{P}^d}(1)$. Then $\mathcal{E} := f_*\mathcal{O}_X(L)$ satisfies $X = \mathbb{P}_Y(\mathcal{E})$ as desired. □

Lemma 2.3 ([2, Remark 2 after Theorem 6]). *Let $p : E \rightarrow B$ be a compact fiber space with fiber F . If the cohomology $H^*(F, \mathbb{Z})$ of fibers is constant and torsion-free, if the fundamental group $\pi_1(B)$ acts trivially on $H^*(F, \mathbb{Z})$, and if $H^+(F, \mathbb{Z}) := \{x \in H^*(F, \mathbb{Z}) \mid \deg(x) > 0\}$ is generated by its elements of minimal positive degree, then the inclusion $F \subset E$ induces a surjection $H^*(E, \mathbb{Z}) \rightarrow H^*(F, \mathbb{Z})$.*

Proposition 2.4 ([15, Lemma 4.1]). *Let X be a Fano manifold admitting a \mathbb{P}^r -bundle structure $f : X \rightarrow Y$ and R the extremal ray corresponding to f . If there exists a proper morphism $g : X \rightarrow Z$ onto a variety Z of dimension r which does not contract curves of R . Then $X \cong \mathbb{P}^r \times Y$.*

Proposition 2.5 ([15, Proposition 5.1]). *Let X be a Fano manifold of dimension n and pseudoindex ≥ 2 which has only contractions of fiber type. Then $\rho_X \leq n$. Moreover,*

- (i) *if $\rho_X = n$, then $X = (\mathbb{P}^1)^n$;*
- (ii) *if $\rho_X = n - 1$, then X is either $(\mathbb{P}^1)^{n-2} \times \mathbb{P}^2$ or $X = (\mathbb{P}^1)^{n-3} \times \mathbb{P}(T_{\mathbb{P}^2})$.*

Proposition 2.6 ([16, Theorem 2]). *Let X be a projective manifold of dimension n , endowed with two different projective bundle structures $f : X \rightarrow Y$ and $g : X \rightarrow Z$ such that $\dim Y + \dim Z = n + 1$. Then either $n = 2m - 1$, $Y = Z = \mathbb{P}^m$ and $X = \mathbb{P}(T_{\mathbb{P}^m})$ or Y and Z have a projective bundle structure over a smooth curve C and $X = Y \times_C Z$.*

3. FANO MANIFOLDS WITH NEF TANGENT BUNDLES

Theorem 3.1 (See [9, Theorem 4.2]). *Let X be a Fano manifold with nef tangent bundle of dimension $n \leq 4$. Then the following holds.*

- (i) *If $n = 1$, then X is \mathbb{P}^1 .*
- (ii) *If $n = 2$, then X is \mathbb{P}^2 or $(\mathbb{P}^1)^2$.*
- (iii) *If $n = 3$, then X is one of the following:
 \mathbb{P}^3 , Q^3 , $\mathbb{P}^1 \times \mathbb{P}^2$, $\mathbb{P}(T_{\mathbb{P}^2})$, $(\mathbb{P}^1)^3$.*
- (iv) *If $n = 4$, then X is one of the following:
 \mathbb{P}^4 , Q^4 , $\mathbb{P}^1 \times \mathbb{P}^3$, $\mathbb{P}^1 \times Q^3$, $(\mathbb{P}^2)^2$, $\mathbb{P}(\mathcal{N})$, where \mathcal{N} is the null-correlation bundle over \mathbb{P}^3 (see Example 3.6 below), $(\mathbb{P}^1)^2 \times \mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}(T_{\mathbb{P}^2})$, $(\mathbb{P}^1)^4$.*

Proof. When $n \leq 2$, it is easy to prove our assertion. When $n = 3$, this is in [4, Theorem 5.1, Theorem 6.1]. Of course, this also follows from the classification theory of Fano manifolds of $n \leq 3$. If $n = 4$ and $\rho_X > 1$, then our assertion is dealt in [5, Theorem 3.1]. However we should remark that the tangent bundle of $\mathbb{P}(T_{\mathbb{P}^2}) \times_{\mathbb{P}^2} \mathbb{P}(T_{\mathbb{P}^2})$, which is listed in [5, Theorem 3.1 (4)-(d)], is not nef, see Lemma 3.3 below. If $n = 4$ and $\rho_X = 1$, we see that X is isomorphic to \mathbb{P}^4 or Q^4 . This follows from [6], [12] and [9, Theorem 4.3] (see also Section 5). \square

Lemma 3.2. *Let X be a Fano manifold with nef tangent bundle. Then the pseudoindex of X is at least 2.*

Proof. Let C be a rational curve on X and $f : \mathbb{P}^1 \rightarrow C \subset X$ its normalization. Since T_X is nef, so is f^*T_X . This implies that $f^*T_X \cong \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i)$, where $a_i \geq 0$. Furthermore we have an injection $\mathcal{O}_{\mathbb{P}^1}(2) \rightarrow f^*T_X$. This implies that $a_i \geq 2$ for some i . Consequently, $-K_X \cdot C = \sum a_i \geq 2$. This means the pseudoindex of X is at least 2. \square

Lemma 3.3. *The tangent bundle of $\mathbb{P}(T_{\mathbb{P}^2}) \times_{\mathbb{P}^2} \mathbb{P}(T_{\mathbb{P}^2})$ is not nef.*

Proof. For $X := \mathbb{P}(T_{\mathbb{P}^2}) \times_{\mathbb{P}^2} \mathbb{P}(T_{\mathbb{P}^2})$, consider the commutative diagram:

$$\begin{array}{ccccc}
 & & X & & \\
 & \swarrow \pi_1 & \downarrow \pi & \searrow \pi_2 & \\
 & \mathbb{P}(T_{\mathbb{P}^2}) & & \mathbb{P}(T_{\mathbb{P}^2}) & \\
 \swarrow p_1 & & \downarrow p_2 & & \searrow p_1 \\
 \mathbb{P}^2 & & \mathbb{P}^2 & & \mathbb{P}^2
 \end{array}$$

Then we have

$$\begin{aligned}
 -K_X &= \pi^*(-K_{\mathbb{P}^2}) + (-K_{X/\mathbb{P}^2}) \\
 &= \pi^*(-K_{\mathbb{P}^2}) + \pi_1^*(-K_{\mathbb{P}(T_{\mathbb{P}^2})/\mathbb{P}^2}) + \pi_2^*(-K_{\mathbb{P}(T_{\mathbb{P}^2})/\mathbb{P}^2}) \\
 &= \pi^*(-K_{\mathbb{P}^2}) + \pi_1^*(-K_{\mathbb{P}(T_{\mathbb{P}^2})} + p_2^*(K_{\mathbb{P}^2})) + \pi_2^*(-K_{\mathbb{P}(T_{\mathbb{P}^2})} + p_2^*(K_{\mathbb{P}^2})) \\
 &= \pi^*(K_{\mathbb{P}^2}) + \pi_1^*(-K_{\mathbb{P}(T_{\mathbb{P}^2})}) + \pi_2^*(-K_{\mathbb{P}(T_{\mathbb{P}^2})}).
 \end{aligned}$$

Let $l \subset \mathbb{P}(T_{\mathbb{P}^2})$ be a fiber of p_1 . Then $p_{2*}(l)$ is a line in \mathbb{P}^2 . Furthermore, l can be regarded as a curve in X via the diagonal embedding $\mathbb{P}(T_{\mathbb{P}^2}) \subset X$. Then

$$-K_X.l = (\pi^*(K_{\mathbb{P}^2}) + \pi_1^*(-K_{\mathbb{P}(T_{\mathbb{P}^2})}) + \pi_2^*(-K_{\mathbb{P}(T_{\mathbb{P}^2})})).l = 1.$$

Thus, Lemma 3.2 concludes that the tangent bundle of X is not nef. \square

Theorem 3.4. *Let X be a Fano manifold with nef tangent bundle, $f : X \rightarrow Y$ a contraction of an extremal ray and X_y a fiber of f . Then the following holds.*

- (i) f is smooth, in particular, of fiber type.
- (ii) Y is a Fano manifold with nef tangent bundle of $\rho_Y = \rho_X - 1$.
- (iii) X_y is a Fano manifold with nef tangent bundle of $\rho_{X_y} = 1$.

Proof. (i) This is in [7, Theorem 5.2] (see also [21, Theorem 4.4]).

(ii) An image of a Fano manifold by a smooth morphism is again Fano (see [11, Corollary 2.9]). Furthermore, since $T_X \rightarrow f^*T_Y$ is surjective, we see that T_Y is nef. Hence (ii) holds.

(iii) From Proposition 2.1, it follows that X_y is a Fano manifold of $\rho_{X_y} = 1$. Moreover we have an exact sequence

$$0 \rightarrow T_{X_y} \rightarrow T_X|_{X_y} \rightarrow N_{X_y/X} \rightarrow 0.$$

Since $c_1(N_{X_y/X}) = 0$, it follows from [4, Proposition 1.8 (8)] that T_{X_y} is nef. \square

Proposition 3.5. *Let X be a Fano manifold with nef tangent bundle, $f : X \rightarrow Y$ a contraction of an extremal ray and F a projective submanifold of Y whose normal bundle is trivial, i.e., $N_{F/Y} \cong \mathcal{O}_F^{\oplus l}$. Then the preimage $W := f^{-1}(F)$ is a Fano manifold with nef tangent bundle.*

Proof. By [8, II. Proposition 8.10], we see that $T_{W/F} \cong T_{X/Y}|_W$. So we have the following exact commutative diagram:

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \uparrow & & \uparrow & \\
 & & & N_{W/X} & & f_W^*(N_{F/Y}) & \\
 & & & \uparrow & & \uparrow & \\
 0 & \longrightarrow & T_{X/Y}|_W & \longrightarrow & T_X|_W & \longrightarrow & f^*(T_Y)|_W \cong f_W^*(T_Y|_F) \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow & \\
 0 & \longrightarrow & T_{W/F} & \longrightarrow & T_W & \longrightarrow & f_W^*(T_F) \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow & \\
 & & 0 & & 0 & & 0 &
 \end{array}$$

Thus the snake lemma implies that $N_{W/X} \cong f_W^*(N_{F/Y})$. By our assumption, we obtain $N_{W/X} \cong \mathcal{O}_W^{\oplus l}$. Then it follows in a similar way to Theorem 3.4 (iii) that T_W is nef. Furthermore, the adjunction formula tells us that $-K_W = (-K_X)|_W$. This means that W is also a Fano manifold. \square

Example 3.6 (Spinor bundle and Null-correlation bundle). Let denote the null-correlation bundle on \mathbb{P}^3 by \mathcal{N} (see [17, Section 4.2] for the definition). Denote by

\mathcal{S} the spinor bundle on Q^3 , by \mathcal{S}_1 and \mathcal{S}_2 the two spinor bundles on Q^4 (see [18, Definition 1.3]).

Then it is known that $\mathbb{P}(\mathcal{N})$ and $\mathbb{P}(\mathcal{S})$ coincides with the full-flag manifold of type B_2 . In particular, $\mathbb{P}(\mathcal{N}) = \mathbb{P}(\mathcal{S})$ is a homogeneous manifold.

On the other hand, the two spinor bundles \mathcal{S}_1 and \mathcal{S}_2 on Q^4 are the universal bundle and the dual of the quotient bundle (see [18, Example 1.5]). Thus, $\mathbb{P}(\mathcal{S}_1)$ and $\mathbb{P}(\mathcal{S}_2)$ are isomorphic to the flag manifold $F(1, 2, \mathbb{P}^3)$ parametrizing pairs (l, P) , where l is a line in a plane $P \subset \mathbb{P}^3$. In particular, $\mathbb{P}(\mathcal{S}_1) \cong \mathbb{P}(\mathcal{S}_2)$ is a homogeneous manifold.

Lemma 3.7. *Let \mathcal{F} be a rank 2 stable vector bundle on Q^4 with Chern classes $c_1 = -1$ and $c_2 = (1, 1)$. Then the tangent bundle of $\mathbb{P}(\mathcal{F})$ is not nef.*

Proof. According to [19, Remark 3.4], \mathcal{F} extends to Q^5 to the Cayley bundle \mathcal{C} . The Cayley bundle is characterized by its Chern class among rank 2 stable bundles on Q^5 (see [19, Main Theorem]). Let $K(G_2)$ be the 5-dimensional contact homogeneous manifold of type G_2 . It is known that $K(G_2)$ is a linear section of the Grassmannian $G(1, \mathbb{P}^6)$ with a \mathbb{P}^{13} . For the restriction of the universal quotient bundle \mathcal{Q} on $G(1, \mathbb{P}^6)$, we see that $\mathbb{P}(\mathcal{Q}|_{K(G_2)})$ coincides with $\mathbb{P}(\mathcal{C})$. Then it follows from [19, 1.3] that $K(G_2)$ is the variety of special lines in Q^5 and $\mathbb{P}(\mathcal{C}) = \mathbb{P}(\mathcal{Q}|_{K(G_2)})$ is its flag variety $\{(p, l) | p \in l, l \text{ special line in } Q^5\}$:

$$\begin{array}{ccc} & \mathbb{P}(\mathcal{C}) = \mathbb{P}(\mathcal{Q}|_{K(G_2)}) & \\ p_1 \swarrow & & \searrow p_2 \\ Q^5 & & K(G_2) \end{array}$$

Since Q^4 is a hyperplane section of Q^5 , the restriction map $p_2|_{\mathbb{P}(\mathcal{F})} : \mathbb{P}(\mathcal{F}) \rightarrow K(G_2)$ is surjective. Furthermore, by [19, Theorem 3.5] and its proof, it turns out that $Q^4 \subset Q^5$ contains a special line l_0 in Q^5 . It implies that $p_2|_{\mathbb{P}(\mathcal{F})}$ has a positive-dimensional fiber. By taking the Stein factorization, one can factor $p_2|_{\mathbb{P}(\mathcal{F})}$ into $g \circ f$, where f is a projective morphism with connected fibers, and g is a finite morphism. Since $p_2|_{\mathbb{P}(\mathcal{F})}$ has a positive-dimensional fiber and $\mathbb{P}(\mathcal{F})$ is a Fano manifold (see [1, Example 2.2]), f is a contraction of an extremal face.

If the tangent bundle of $\mathbb{P}(\mathcal{F})$ would be nef, then it follows from Theorem 3.4 that f is of fiber type. However it contradicts to $\dim \mathbb{P}(\mathcal{F}) = \dim K(G_2)$. \square

Proposition 3.8. *Let X be a Fano manifold with nef tangent bundle which admits a \mathbb{P}^1 -bundle structure $f : X \rightarrow Y$. Let \mathcal{N} be the null-correlation bundle on \mathbb{P}^3 , \mathcal{S} the spinor bundle on Q^3 and \mathcal{S}_i ($i = 1, 2$) the spinor bundles on Q^4 as in Example 3.6. Then the following holds.*

- (i) *If Y is \mathbb{P}^4 , then X is $\mathbb{P}^1 \times \mathbb{P}^4$.*
- (ii) *If Y is Q^4 , then X is $\mathbb{P}^1 \times Q^4$ or $\mathbb{P}(\mathcal{S}_i)$.*
- (iii) *If Y is $\mathbb{P}^1 \times \mathbb{P}^3$ (resp. $\mathbb{P}^1 \times Q^3$), then X is $(\mathbb{P}^1)^2 \times \mathbb{P}^3$ or $\mathbb{P}^1 \times \mathbb{P}(\mathcal{N})$ (resp. $(\mathbb{P}^1)^2 \times Q^3$ or $\mathbb{P}^1 \times \mathbb{P}(\mathcal{S})$).*
- (iv) *If Y is $\mathbb{P}(\mathcal{N})$, then X is $\mathbb{P}^1 \times \mathbb{P}(\mathcal{N})$.*
- (v) *If Y is $(\mathbb{P}^2)^2$, then X is $\mathbb{P}^1 \times (\mathbb{P}^2)^2$ or $\mathbb{P}^2 \times \mathbb{P}(T_{\mathbb{P}^2})$.*

In particular, every manifold appeared in the above list is rational homogeneous.

Proof. Let \mathcal{E} be a rank 2 vector bundle on Y such that $X = \mathbb{P}(\mathcal{E})$.

(i) If Y is \mathbb{P}^4 , then it follows from [1, Main Theorem 2.4] that \mathcal{E} splits into a direct sum of line bundles as $\mathcal{O}_Y(a) \oplus \mathcal{O}_Y(b)$. If a is not equal to b , then Y has a contraction of birational type. However this contradicts to Theorem 3.4 (i). Hence X is $\mathbb{P}^1 \times \mathbb{P}^4$.

(ii) If Y is Q^4 , then [1, Main Theorem 2.4] and Lemma 3.7 imply that X is $\mathbb{P}^1 \times Q^4$ or $\mathbb{P}(\mathcal{S}_i)$, via the same argument as in (i).

(iii) Let Y be $\mathbb{P}^1 \times V$, where V is \mathbb{P}^3 or Q^3 . Let p_1 be the first projection $Y \rightarrow \mathbb{P}^1$ and p_2 the second projection $Y \rightarrow V$:

$$\begin{array}{ccc} & Y & \\ p_1 \swarrow & & \searrow p_2 \\ \mathbb{P}^1 & & V \end{array}$$

Let l be a fiber of p_2 . According to Proposition 3.5, $\mathbb{P}(\mathcal{E}|_l)$ is a Fano surface with nef tangent bundle. Thus, by Theorem 3.1, we see that $\mathcal{E}|_l \cong \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}$ up to a twist by a line bundle. Thus, by tensoring a line bundle, we may assume that $\mathcal{E}|_l \cong \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}$ for every fiber l of p_2 . By applying Grauert's theorem [8, III. Corollary 12.9], we see that $p_{2*}(\mathcal{E})$ is a rank 2 vector bundle on V . Furthermore, there is a natural map $p_2^*(p_{2*}(\mathcal{E})) \rightarrow \mathcal{E}$. For $y \in l$, we have $p_2^*(p_{2*}(\mathcal{E})) \otimes k(y) \cong H^0(l, \mathcal{E}|_l)$. Again, this follows from Grauert's theorem [8, III. Corollary 12.9]. Hence $p_2^*(p_{2*}(\mathcal{E})) \otimes k(y) \rightarrow \mathcal{E} \otimes k(y)$ is surjective. By Nakayama's lemma, $p_2^*(p_{2*}(\mathcal{E}))_y \rightarrow \mathcal{E}_y$ is also surjective, hence, so is $p_2^*(p_{2*}(\mathcal{E})) \rightarrow \mathcal{E}$. As a consequence, it turns out that

$$p_2^*(p_{2*}(\mathcal{E})) \cong \mathcal{E}.$$

For a fiber F of p_1 , $p_{2*}(\mathcal{E}) \cong p_2^*(p_{2*}(\mathcal{E}))|_F \cong \mathcal{E}|_F$. This implies that $\mathcal{E} \cong p_2^*(p_{2*}(\mathcal{E})) \cong p_2^*(\mathcal{E}|_F)$. Thus, we see that $X \cong \mathbb{P}^1 \times \mathbb{P}(\mathcal{E}|_F)$. By Proposition 3.5, $\mathbb{P}(\mathcal{E}|_F)$ is a Fano 4-fold with nef tangent bundle. According to Theorem 3.1, if $F \cong \mathbb{P}^3$ (resp. $F \cong Q^3$), then $\mathbb{P}(\mathcal{E}|_F)$ is $\mathbb{P}^1 \times \mathbb{P}^3$ or $\mathbb{P}(\mathcal{N})$ (resp. $\mathbb{P}^1 \times Q^3$ or $\mathbb{P}(\mathcal{S})$). Hence our assertion holds.

(iv) Let Y be $\mathbb{P}(\mathcal{N})$ and $p : \mathbb{P}(\mathcal{N}) \rightarrow \mathbb{P}^3$ the bundle projection. By a similar argument to (iii), one can show that $\mathcal{E}|_l \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ for a fiber l of p , and $\mathcal{E} = p^*(\mathcal{E}_0)$ for $\mathcal{E}_0 := p_*(\mathcal{E})$. Now we have a base change diagram

$$\begin{array}{ccc} \mathbb{P}(\mathcal{E}) & \longrightarrow & \mathbb{P}(\mathcal{E}_0) \\ \downarrow & & \downarrow \\ \mathbb{P}(\mathcal{N}) & \xrightarrow{p} & \mathbb{P}^3 \end{array}$$

Since $X = \mathbb{P}(\mathcal{E})$ is a \mathbb{P}^1 -bundle over $\mathbb{P}(\mathcal{E}_0)$, $\mathbb{P}(\mathcal{E}_0)$ is a Fano 4-fold with nef tangent bundle. Moreover $\mathbb{P}(\mathcal{E}_0)$ is a \mathbb{P}^1 -bundle over \mathbb{P}^3 . Thus, by Theorem 3.1, $\mathbb{P}(\mathcal{E}_0)$ is $\mathbb{P}^1 \times \mathbb{P}^3$ or $\mathbb{P}(\mathcal{N})$. This implies that X is $\mathbb{P}^1 \times \mathbb{P}(\mathcal{N})$ or $\mathbb{P}(\mathcal{N}) \times_{\mathbb{P}^3} \mathbb{P}(\mathcal{N})$. In the later case, we can show that the tangent bundle of X is not nef in a similar way to Lemma 3.3. Indeed, $\mathbb{P}(\mathcal{N})$ admits a \mathbb{P}^1 -bundle structure over Q^3 and denote its fiber by l . Remark that l can be regarded as a curve in $X := \mathbb{P}(\mathcal{N}) \times_{\mathbb{P}^3} \mathbb{P}(\mathcal{N})$ via the diagonal embedding $\mathbb{P}(\mathcal{N}) \subset X$. Then we see that $-K_X.l = 0$. This implies that X is not Fano. Hence our assertion holds.

(v) Let Y be $(\mathbb{P}^2)^2$ and p_i the i -th projection $Y \rightarrow \mathbb{P}^2$ ($i = 1, 2$). Let F_i be a fiber of p_i . According to Proposition 3.5, $\mathbb{P}(\mathcal{E}|_{F_i})$ is a Fano manifold with nef

tangent bundle. Thus, by Theorem 3.1, we see that $\mathcal{E}|_{F_i} \cong \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}$ or $T_{\mathbb{P}^2}(-1)$, up to a twist by a line bundle. If $\mathcal{E}|_{F_i} \cong \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}$ for some i , then we see that $p_i^*(p_{i*}(\mathcal{E})) \cong \mathcal{E}$ in a similar way to (iii). Furthermore, $\mathcal{E} \cong p_i^*(\mathcal{E}|_{F_j})$ and $\mathcal{E}|_{F_j} \cong \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}$ or $T_{\mathbb{P}^2}(-1)$ for $j \neq i$. As a consequence, X is $\mathbb{P}^1 \times (\mathbb{P}^2)^2$ or $\mathbb{P}^2 \times \mathbb{P}(T_{\mathbb{P}^2})$. On the other hand, assume that $\mathcal{E}|_{F_i} \cong T_{\mathbb{P}^2}(-1)$ for $i = 1, 2$. Then $c_1(\mathcal{E}) = (1, 1)$. This implies that $\mathcal{O}_X(-K_X) \cong \mathcal{O}_{\mathbb{P}(\mathcal{E})}(2) \otimes f^*\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(2, 2)$, where $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is the tautological invertible sheaf of $X = \mathbb{P}(\mathcal{E})$. This implies that the Fano index of X is 2. According to Theorem 3.4, X has only contractions of fiber type. Thus, it follows from [15, Proposition 7.1] that X is a product with \mathbb{P}^1 as a factor. However, this contradicts to $\mathcal{E}|_{F_i} \cong T_{\mathbb{P}^2}(-1)$. \square

Proposition 3.9. *Let X be a Fano 5-fold with nef tangent bundle. Then $\rho_X \leq 3$ or X is one of the following:
 $(\mathbb{P}^1)^5$, $(\mathbb{P}^1)^3 \times \mathbb{P}^2$, $(\mathbb{P}^1)^2 \times \mathbb{P}(T_{\mathbb{P}^2})$.*

Proof. By Lemma 3.2, the pseudoindex of X is at least 2. Moreover, X has only contractions of fiber type because of Theorem 3.4. Thus, by applying Proposition 2.5, we get our assertion. \square

4. PROOF OF THEOREM 1.2

Let \mathcal{N} be the null-correlation bundle on \mathbb{P}^3 , \mathcal{S} the spinor bundle on Q^3 and \mathcal{S}_i ($i = 1, 2$) the spinor bundles on Q^4 as in Example 3.6. In this section, we prove Theorem 1.2:

Theorem 4.1 (=Theorem 1.2). *Let X be a Fano manifold of dimension 5 with nef tangent bundle and Picard number $\rho_X > 1$. Then X is one of the following:
 $\mathbb{P}^1 \times \mathbb{P}^4$, $\mathbb{P}^1 \times Q^4$, $\mathbb{P}^2 \times \mathbb{P}^3$, $\mathbb{P}^2 \times Q^3$, $\mathbb{P}(T_{\mathbb{P}^3})$, $\mathbb{P}(\mathcal{S}_i)$, $\mathbb{P}^1 \times (\mathbb{P}^2)^2$, $(\mathbb{P}^1)^2 \times \mathbb{P}^3$, $(\mathbb{P}^1)^2 \times Q^3$, $\mathbb{P}^2 \times \mathbb{P}(T_{\mathbb{P}^2})$, $\mathbb{P}^1 \times \mathbb{P}(\mathcal{N}) = \mathbb{P}^1 \times \mathbb{P}(\mathcal{S})$, $(\mathbb{P}^1)^3 \times \mathbb{P}^2$, $(\mathbb{P}^1)^2 \times \mathbb{P}(T_{\mathbb{P}^2})$, $(\mathbb{P}^1)^5$.*

In particular, X is a rational homogeneous manifold.

Let X be a Fano 5-fold with nef tangent bundle of $\rho_X \geq 2$. Then there exist two different contractions $f : X \rightarrow Y$ and $g : X \rightarrow Z$ of extremal rays:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ g \downarrow & & \\ Z & & \end{array}$$

Denote by X_y (resp. X_z) a fiber of f (resp. one of g). We may assume that $\dim Z \geq \dim Y (\geq 1)$.

Lemma 4.2. *Under the above setting, the following holds.*

- (i) $\rho_Y = \rho_Z$.
- (ii) Y and Z are rational homogeneous manifolds listed in Theorem 3.1. Furthermore, X_y and X_z are either \mathbb{P}^d ($1 \leq d \leq 4$) or Q^d ($d = 3$ or 4).
- (iii) $5 > \dim Y \geq \dim X_z$ and $5 > \dim Z \geq \dim X_y$.
- (iv) If $\dim Z = \dim X_y$ and $X_y \cong \mathbb{P}^d$ (resp. $\dim Y = \dim X_z$ and $X_z \cong \mathbb{P}^d$), then we have $X \cong \mathbb{P}^d \times Y$ (resp. $\mathbb{P}^d \times Z$).

- (v) If $\dim Z = \dim X_y$ and $X_y \cong Q^d$ ($d = 3$ or 4) (resp. $\dim Y = \dim X_z$ and $X_z \cong Q^d$), then Z (resp. Y) is either \mathbb{P}^d or Q^d and X is a \mathbb{P}^{5-d} -bundle over Z (resp. Y).

Proof. (i) Since f and g are contractions of extremal rays, $\rho_Y = \rho_X - 1 = \rho_Z$.

(ii) From Theorem 3.4, Y , Z , X_y and X_z are Fano manifolds with nef tangent bundles, and $\rho_{X_y} = \rho_{X_z} = 1$. Hence Theorem 3.1 implies our assertion.

(iii) Since f and g are different contractions, X_y and X_z are not contracted by g and f , respectively. Furthermore, we have $\rho_{X_y} = \rho_{X_z} = 1$. This implies that $\dim Y \geq \dim X_z$ and $\dim Z \geq \dim X_y$.

(iv) If $\dim Z = \dim X_y$ and $X_y \cong \mathbb{P}^d$, then our claim follows from Proposition 2.2 and Proposition 2.4.

(v) We see that $Z \cong \mathbb{P}^d$ or Q^d by [20, Proposition 8], and it follows from (ii) and Proposition 2.2 that X is a \mathbb{P}^{5-d} -bundle over Z . \square

4.1. Case where $\dim Y = 1$.

Proposition 4.3. *If $\dim Y = 1$, then X is $\mathbb{P}^1 \times \mathbb{P}^4$ or $\mathbb{P}^1 \times Q^4$.*

Proof. By Lemma 4.2 (ii), $Y \cong \mathbb{P}^1$ and $X_y \cong \mathbb{P}^4$ or Q^4 . Furthermore, it follows from Lemma 4.2 (iii) that $\dim Z = \dim X_y = 4$. If $X_y \cong \mathbb{P}^4$, then Lemma 4.2 (iv) concludes that $X \cong \mathbb{P}^1 \times \mathbb{P}^4$. On the other hand, if $X_y \cong Q^4$, then Lemma 4.2 (v) tells us that X is a \mathbb{P}^1 -bundle over Z . Then, using Proposition 2.4, we see that $X \cong \mathbb{P}^1 \times Q^4$. \square

4.2. Case where $\dim Y = 2$.

Proposition 4.4. *If $\dim Y = 2$, then $X \cong \mathbb{P}^2 \times \mathbb{P}^3$, $\mathbb{P}^2 \times Q^3$, $(\mathbb{P}^1)^2 \times \mathbb{P}^3$ or $(\mathbb{P}^1)^2 \times Q^3$.*

Proof. By Lemma 4.2, we see that $Y \cong \mathbb{P}^2$ or $(\mathbb{P}^1)^2$, $X_y \cong \mathbb{P}^3$ or Q^3 and $\dim Z = 3$ or 4 , in a similar way to Proposition 4.3.

If $Y \cong \mathbb{P}^2$ and $\dim Z = 3$, then we have $\dim Y = \dim X_z$ and it follows from Lemma 4.2 (ii) that $X_z \cong \mathbb{P}^2$. Therefore Lemma 4.2 (iv) implies that $X \cong \mathbb{P}^2 \times \mathbb{P}^3$ or $\mathbb{P}^2 \times Q^3$.

If $Y \cong (\mathbb{P}^1)^2$ and $\dim Z = 4$, then X is a \mathbb{P}^1 -bundle over \mathbb{P}^4 or Q^4 by Lemma 4.2 (ii) and Proposition 2.2. Therefore we are in the situation of Proposition 3.8 (i) and (ii). However every manifold appeared there has no contractions to \mathbb{P}^2 . Hence we get a contradiction.

If $Y \cong (\mathbb{P}^1)^2$, then it follows from Lemma 4.2 (i) that $\rho_Z = \rho_Y = 2$. By virtue of Lemma 4.2 (ii), $X_y \cong \mathbb{P}^3$ or Q^3 . If $\dim Z = 3$, then Z would be isomorphic to \mathbb{P}^3 or Q^3 by Lemma 4.2 (iv) and (v). This contradicts to $\rho_Z = 2$. Hence $\dim Z = 4$. Then, it follows from Lemma 4.2 (ii) and Proposition 2.2 that X is a \mathbb{P}^1 -bundle over Z , where $Z \cong \mathbb{P}^1 \times \mathbb{P}^3$, $\mathbb{P}^1 \times Q^3$, $(\mathbb{P}^1)^2$ or $\mathbb{P}(\mathcal{N})$. Thus we are in the situation of Proposition 3.8 (iii) – (v). Since X admits a contraction of an extremal ray to $Y \cong (\mathbb{P}^1)^2$, we see that $X \cong (\mathbb{P}^1)^2 \times \mathbb{P}^3$ or $(\mathbb{P}^1)^2 \times Q^3$. \square

4.3. Case where $\dim Y = 3$.

Proposition 4.5. *If $\dim Y = 3$, then $X \cong \mathbb{P}(T_{\mathbb{P}^3})$, $\mathbb{P}(\mathcal{S}_i)$, $\mathbb{P}^1 \times (\mathbb{P}^2)^2$, $\mathbb{P}^2 \times \mathbb{P}(T_{\mathbb{P}^2})$ or $(\mathbb{P}^1)^3 \times \mathbb{P}^2$.*

Proof. According to Proposition 3.9, $\rho_X \leq 3$ if X is not isomorphic to $(\mathbb{P}^1)^5$, $(\mathbb{P}^1)^3 \times \mathbb{P}^2$ or $(\mathbb{P}^1)^2 \times \mathbb{P}(T_{\mathbb{P}^2})$. Since $(\mathbb{P}^1)^5$ and $(\mathbb{P}^1)^2 \times \mathbb{P}(T_{\mathbb{P}^2})$ have no contractions of extremal rays to 3-dimensional manifolds, we have $X \cong (\mathbb{P}^1)^3 \times \mathbb{P}^2$ or $\rho_X \leq 3$. So it is enough to consider the case where $\rho_X \leq 3$. Then it follows from Lemma 4.2 (i) that $\rho_Y = \rho_Z \leq 2$. By our assumption, we see that $5 > \dim Z \geq \dim Y = 3$.

If $\dim Z = 3$, then it follows from Lemma 4.2 (ii) and Proposition 2.2 that X admits two different \mathbb{P}^2 -bundle structures. By Proposition 2.6, $X = \mathbb{P}(T_{\mathbb{P}^3})$ or $Y \times_C Z$, where Y and Z are \mathbb{P}^2 -bundles over a smooth curve C . In the latter case, since Y and Z are projective bundles of $\rho = 2$, it follows from Theorem 3.1 (iii) that $Y \cong Z \cong \mathbb{P}^1 \times \mathbb{P}^2$ and $C \cong \mathbb{P}^1$. Therefore, $X \cong (\mathbb{P}^1 \times \mathbb{P}^2) \times_{\mathbb{P}^1} (\mathbb{P}^1 \times \mathbb{P}^2) \cong \mathbb{P}^1 \times (\mathbb{P}^2)^2$.

If $\dim Z = 4$, then Lemma 4.2 (ii) and Proposition 2.2 imply that X is a \mathbb{P}^1 -bundle over \mathbb{P}^4 , Q^4 , $\mathbb{P}^1 \times \mathbb{P}^3$, $\mathbb{P}^1 \times Q^3$, $(\mathbb{P}^2)^2$ or $\mathbb{P}(\mathcal{N})$. Therefore we are in the situation of Proposition 3.8 (i) – (v). Since X admits a contraction of an extremal ray to a 3-dimensional manifold Y , X is $\mathbb{P}(\mathcal{S}_i)$, $\mathbb{P}^1 \times (\mathbb{P}^2)^2$ or $\mathbb{P}^2 \times \mathbb{P}(T_{\mathbb{P}^2})$. □

4.4. Case where $\dim Y = 4$.

Proposition 4.6. *If $\dim Y = 4$, then X is isomorphic to one of the following: $(\mathbb{P}^1)^5$, $(\mathbb{P}^1)^3 \times \mathbb{P}^2$, $(\mathbb{P}^1)^2 \times \mathbb{P}(T_{\mathbb{P}^2})$, $(\mathbb{P}^1)^2 \times \mathbb{P}^3$, $(\mathbb{P}^1)^2 \times Q^3$, $\mathbb{P}^1 \times \mathbb{P}(\mathcal{N}) = \mathbb{P}^1 \times \mathbb{P}(\mathcal{S})$, $\mathbb{P}^2 \times \mathbb{P}(T_{\mathbb{P}^2})$.*

Proof. According to Proposition 3.9, $\rho_X \leq 3$ if X is not isomorphic to $(\mathbb{P}^1)^5$, $(\mathbb{P}^1)^3 \times \mathbb{P}^2$ or $(\mathbb{P}^1)^2 \times \mathbb{P}(T_{\mathbb{P}^2})$. So it is enough to consider the case where $\rho_X \leq 3$. Then it is equivalent to $\rho_Y = \rho_Z \leq 2$. Lemma 4.2 (ii) and Proposition 2.2 imply that X admits two different \mathbb{P}^1 -bundle structures over 4-folds Y and Z of $\rho \leq 2$. By Lemma 4.2 (ii), Y and Z are \mathbb{P}^4 , Q^4 , $\mathbb{P}^1 \times \mathbb{P}^3$, $\mathbb{P}^1 \times Q^3$, $(\mathbb{P}^2)^2$ or $\mathbb{P}(\mathcal{N})$. Therefore we are in the situation of Proposition 3.8 (i) – (v). Since X admits two different \mathbb{P}^1 -bundle structures over 4-folds Y and Z of $\rho \leq 2$, X is $(\mathbb{P}^1)^2 \times \mathbb{P}^3$, $(\mathbb{P}^1)^2 \times Q^3$, $\mathbb{P}^1 \times \mathbb{P}(\mathcal{N}) = \mathbb{P}^1 \times \mathbb{P}(\mathcal{S})$ or $\mathbb{P}^2 \times \mathbb{P}(T_{\mathbb{P}^2})$. □

5. CASE WHERE $\rho_X = 1$

Finally, we deal with Fano manifolds with nef tangent bundles of $\rho_X = 1$. All the results in this section are well-known for experts.

Theorem 5.1. *Let X be a smooth Fano n -fold with nef tangent bundle of $\rho_X = 1$. Then the pseudoindex i_X satisfies $3 \leq i_X \leq n + 1$. Furthermore, the following holds.*

- (i) *If $i_X = n + 1$, then X is \mathbb{P}^n .*
- (ii) *If $i_X = n$, then X is Q^n .*
- (iii) *If $i_X = 3$, then X is \mathbb{P}^2 , Q^3 or $K(G_2)$, where $K(G_2)$ is the 5-dimensional contact homogeneous manifold of type G_2 .*

Proof. By virtue of Lemma 3.2, we see that $2 \leq i_X$. Furthermore, it follows from the argument as in [9, Before Theorem 4.3, P. 623] that i_X is not 2. On the other hand, if $i_X \geq n + 1$, then X is \mathbb{P}^n . This is dealt in [6]. If $i_X = n$, then our assertion follows from [12]. The case where $i_X = 3$ is treated in [9, Theorem 4.3]. □

As a consequence, we have the following:

Corollary 5.2. *Let X be a smooth Fano 5-fold with nef tangent bundle of $\rho_X = 1$. Then one of the following holds.*

- (i) X is \mathbb{P}^5 , Q^5 or $K(G_2)$.
- (ii) $i_X = 4$

Remark 5.3. Let X be a smooth Fano 5-fold with nef tangent bundle of $\rho_X = 1$. For the ample generator H of $\text{Pic}(X)$, if there exists a rational curve l such that $H.l = 1$, then we see that the Fano index coincides with the pseudoindex $i_X = 4$. Hence, it turns out that X is a Fano 5-fold with index 4. In other words, X is a del Pezzo 5-fold.

On the other hand, a rational homogeneous manifold of $\rho = 1$ contains a line (see for instance [10, V.1.15]). Furthermore, we see that there is no rational homogeneous 5-fold of $\rho = 1$ with $i_X = 4$.

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